Finite Math - Fall 2018

Lecture Notes - 11/8/2018

Homework

- Section 4.6 - 9, 12, 14, 15, 17, 18, 21, 24, 26, 29, 32, 39, 40, 41, 42, 43, 44, 45, 46, 55, 56
- Section 4.7 - 21, 22, 23, 35, 36, 37, 38

Section 4.6 - Matrix Equations and Systems of Linear Equations

Matrix Equations.

Theorem 1. *Assume that all products and sums are defined for the indicated matrices A, B, C, I, and 0 (where 0 stands for the zero matrix). Then*

- *Addition Properties*

  (1) *Associative*

  $$(A + B) + C = A + (B + C)$$

  (2) *Commutative*

  $$A + B = B + A$$

  (3) *Additive Identity*

  $$A + 0 = 0 + A = A$$

  (4) *Additive Inverse*

  $$A + (-A) = (-A) + A = 0$$

- *Multiplication Properties*

  (1) *Associative Property*

  $$A(BC) = (AB)C$$

  (2) *Multiplicative Identity*

  $$AI = IA = A$$

  (3) *Multiplicative Inverse*
  If A is a square matrix and A^{-1} exists, then $AA^{-1} = A^{-1}A = I$

- *Combined Properties*

  (1) *Left Distributive*

  $$A(B + C) = AB + AC$$

  (2) *Right Distributive*

  $$(B + C)A = BA + CA$$

- *Equality*

  (1) *Addition*
  If $A = B$, then $A + C = B + C$

  (2) *Left Multiplication*
  If $A = B$, then $CA = CB$

  (3) *Right Multiplication*
  If $A = B$, then $AC = BC$

We can use the rules above to solve various matrix equations. In the next 3 examples, we will assume all necessary inverses exists.

Example 1. *Suppose A is an $n \times n$ matrix and B and X are $n \times 1$ column matrices. Solve the matrix equation for X*

$$AX = B.$$

Solution. *If we multiply both sides of this equation ON THE LEFT by A^{-1} we find*

$$A^{-1}(AX) = A^{-1}B \quad \Longrightarrow \quad (A^{-1}A)X = IX = X = A^{-1}B$$

Example 2. *Suppose A is an $n \times n$ matrix and B, C, and X are $n \times 1$ matrices. Solve the matrix equation for X*

$$AX + C = B.$$

Solution. *Begin by subtracting C to the other side*

$$AX + C = B \quad \Longrightarrow \quad AX = B - C$$

and now multiply on the left by A^{-1}

$$A^{-1}(AX) = A^{-1}(B-C) \quad \Longrightarrow \quad (A^{-1}A)X = IX = X = A^{-1}(B-C) = A^{-1}B-A^{-1}C$$

Example 3. *Suppose A and B are $n \times n$ matrices and C is an $n \times 1$ matrix. Solve the matrix equation for X*

$$AX - BX = C.$$

What size matrix is X?

Matrix Equations and Systems of Linear Equations. We can also solve systems of equations using the above ideas. These apply in the case that the system has the same number of variables as equations and the coefficient matrix of the system is invertible. If that is the case, for the system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & a_{1n}x_n & = & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}$$

We can create the matrix equation

$$AX = B$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{1n} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Then, if A is invertible (as is the case when the system is consistent and independent, i.e., exactly one solution), we have

$$X = A^{-1}B.$$

Example 4. *Solve the system of equations using matrix methods*

$$\begin{array}{ccccc} x & + & 2y & = & k_1 \\ x & + & 3y & = & k_2 \end{array}$$

where

(a) $k_1 = 1,\ k_2 = 3$

(b) $k_1 = 3,\ k_2 = 5$

(c) $k_1 = -2,\ k_2 = 1$

Solution. *Begin by writing this system as a matrix equation*

$$AX = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = B$$

Our goal is to find $A^{-1}B$, so first find A^{-1}:

$$A^{-1} = \frac{1}{3-2} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Then

$$X = A^{-1}B = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 3k_1 - 2k_2 \\ -k_1 + k_2 \end{bmatrix}.$$

So, to get the solutions for the different parts, just plug in the given k_1 and k_2's:

(a) $x = -3$ *and* $y = 2$

(b) $x = -1$ *and* $y = 2$

(c) $x = -8$ *and* $y = 3$

Example 5. *Solve the system of equations using matrix methods*

$$\begin{array}{rcrcl} 2x & + & y & = & k_1 \\ 5x & + & 3y & = & k_2 \end{array}$$

where

(a) $k_1 = 2,\ k_2 = 13$

(b) $k_1 = 2,\ k_2 = 4$

(c) $k_1 = 1,\ k_2 = -3$

Solution.

(a) $x = -7$ *and* $y = 16$

(b) $x = 2$ *and* $y = -2$

(c) $x = 6$ *and* $y = -11$

Section 4.7 - Leontief Input-Output Analysis

Two-Industry Model. To simplify the ideas, we will assume we are in an economy with only two industries: coal and steel. In this economy, to produce \$1 worth of coal requires an input of \$0.10 from the coal sector and \$0.20 from the steel sector; and to produce \$1 worth of steel requires an input of \$0.20 from the coal sector and \$0.40 from the steel sector. The final demand (the demand from all other users of coal and steel) is \$20 billion for coal and \$10 billion for steel. What we would like to know is how much total coal and steel needs to be produced to meet this final demand.

If the coal and steel sector produces just the final demand of coal and steel, it would require:

<u>Coal</u>

$$0.1(20) + 0.2(10) = \$4 \text{ billion of coal}$$

<u>Steel</u>

$$0.2(20) + 0.4(10) = \$8 \text{ billion of steel}$$

This only leaves \$16 billion of coal and \$2 billion of steel left over to meet that final demand, well below the required amounts.

So, we need to not only meet the final demand, but also the internal demand. To figure out how to do this, we need two variables

$$x = \text{total output from coal industry}$$

$$y = \text{total output from steel industry.}$$

Then the internal demands (amount of coal and steel required to produce x amount of coal and y amount of steel) are as follows:

Coal

$$0.1x + 0.2y \text{ internal demand for coal}$$

Steel

$$0.2x + 0.4y \text{ internal demand for steel}$$

So, we can create equations for the total amount of coal and steel required by adding the internal and final demands to get the system of equations:

$$\begin{array}{rcccl} \text{Total} & & \text{Internal} & & \text{Final} \\ \text{output} & & \text{demand} & & \text{demand} \\ x & = & 0.1x + 0.2y & + & 20 \\ y & = & 0.2x + 0.4y & + & 10 \end{array}$$

which we rewrite in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

or using letters for the matrices

$$X = MX + D.$$

D is called the *final demand matrix*, X is called the *output matrix*, and M is called the *technology matrix*. The technology matrix should be read as the inputs entering from the left and outputs leaving from above. That is,

$$\begin{array}{ccc} & & \text{Output} \\ & & \begin{array}{cc} C & S \\ \uparrow & \uparrow \end{array} \\ \text{Input} & \begin{array}{c} C \rightarrow \\ \\ S \rightarrow \end{array} & \begin{bmatrix} \begin{pmatrix} \text{input from } C \\ \text{to produce \$1} \\ \text{of coal} \end{pmatrix} & \begin{pmatrix} \text{input from } C \\ \text{to produce \$1} \\ \text{of steel} \end{pmatrix} \\ \begin{pmatrix} \text{input from } S \\ \text{to produce \$1} \\ \text{of coal} \end{pmatrix} & \begin{pmatrix} \text{input from } S \\ \text{to produce \$1} \\ \text{of steel} \end{pmatrix} \end{bmatrix} = M \end{array}$$

(C stands for the coal industry and S for the steel industry).

Now that the individual pieces are understood, let's finish solving the problem. Let's begin by solving the matrix equation first:

$$\begin{aligned} X &= MX + D \\ X - MX &= D \\ (I-M)X &= D \\ X &= (I-M)^{-1}D \end{aligned}$$

(Note that this solution requires $I - M$ to have an inverse!)

Now we actually work this out with the numbers from this problem

Solution.

$$\begin{aligned} I - M &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.9 & -0.2 \\ -0.2 & 0.6 \end{bmatrix} \\ (I-M)^{-1} &= \frac{1}{(0.9)(0.6) - (-0.2)(-0.2)} \begin{bmatrix} 0.6 & 0.2 \\ 0.2 & 0.9 \end{bmatrix} = \begin{bmatrix} 1.2 & 0.4 \\ 0.4 & 1.8 \end{bmatrix} \\ X = (I-M)^{-1}D &= \begin{bmatrix} 1.2 & 0.4 \\ 0.4 & 1.8 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 28 \\ 26 \end{bmatrix} \end{aligned}$$

So to meet the internal and final demands, \$28 *billion of coal and* \$26 *billion of steel must be produced.*

To summarize, these are the steps to solving an input-output analysis problem:

(1) Find the technology matrix M and the final demand matrix D.

(2) Find $I - M$.

(3) Find $(I-M)^{-1}$.

(4) Find $X = (I-M)^{-1}D$.

(5) Interpret the answer in words.